

QUASIPERIODIC AND MIXED COMMUTATOR FACTORIZATIONS IN FREE PRODUCTS OF GROUPS

S. V. IVANOV AND A. A. KLYACHKO

ABSTRACT. It is well known that a nontrivial commutator in a free group is never a proper power. We prove a theorem that generalizes this fact and has several worthwhile corollaries. For example, an equation $[x_1, y_1] \dots [x_k, y_k] = z^n$, where $n \geq 2k$, in a free product \mathcal{F} of groups without nontrivial elements of order $\leq n$ implies that z is conjugate to an element of a free factor of \mathcal{F} . If a nontrivial commutator in a free group factors into a product of elements which are conjugate to each other then all these elements are distinct.

1. INTRODUCTION

It was observed by Schützenberger [31] that a nontrivial commutator in a free group is never a proper power. This result was generalized in different directions: for values of other than commutator words on free groups by Baumslag and Steinberg [1], for values of commutators on free products of groups by Comerford, Edmunds, and Rosenberger [7], and for values of commutators on small cancellation groups by Frenkel and the second author [12]. Our Theorem could be considered as one more such a generalization.

Theorem. *Let G_α , $\alpha \in I$, be torsion free groups and let $\mathcal{F} = \ast_{\alpha \in I} G_\alpha$ denote their free product. Suppose that*

$$c_1 \dots c_k d_1 \dots d_\ell = h_1^{n_1} \dots h_m^{n_m}$$

in \mathcal{F} , where $c_1, \dots, c_k, d_1, \dots, d_\ell, h_1, \dots, h_m$ are elements of \mathcal{F} such that c_i are commutators, d_i are conjugate to elements of free factors of \mathcal{F} , h_j are conjugate to each other and are not conjugate to elements of free factors of \mathcal{F} , and n_j are positive integers. Then

$$\sum_{j=1}^m (n_j - 1) \leq 2k + \ell - 2.$$

Moreover, the same statement holds for a free product \mathcal{F} of groups with torsion whenever the orders of nontrivial elements of free factors of \mathcal{F} are greater than $\sum_{j=1}^m n_j$.

M. Culler [8] discovered that, in the free group $F(a, b)$ with free generators a, b , the cube $[a, b]^3$ of the commutator $[a, b] := a^{-1}b^{-1}ab$ is a product of two commutators,

$$[a, b]^3 = [a^{-1}ba, a^{-2}bab^{-1}][bab^{-1}, b^2].$$

Moreover, $[a, b]^n$ is a product of k commutators whenever $n \leq 2k - 1$, see [8].

2010 *Mathematics Subject Classification.* Primary 20E06, 20F06, 20F70, 57M07.

The work of the second author was supported by the Russian Foundation for Basic Research, project 15-01-05823.

Comerford, Comerford, and Edmunds [6] proved that a nontrivial product of two commutators in a free group cannot be more than a third power, i.e., the equality $[x_1, y_1][x_2, y_2] = z^n$, where $n \geq 4$, in a free group implies that $z = 1$.

The authors of [6] conjectured that, in a free group, the Culler's bound $n \leq 2k - 1$ is sharp. In other words, the Comerford-Comerford-Edmunds conjecture asserts that, in a free group, the equality $[x_1, y_1] \dots [x_k, y_k] = z^n$, where $n \geq 2k$, implies that $z = 1$. This conjecture was proven by Duncan and Howie [9, Theorem 3.3] by establishing that, in a free product $A * B$ of two locally indicable groups A, B , the equality $[x_1, y_1] \dots [x_k, y_k] = z^n$, where $n \geq 2k$, implies that z is conjugate to an element of A or B . Our Theorem implies, in particular, that a similar result holds true for a free product of torsion free groups and for a free product of groups with torsion bounded below. Note that, in the case of small torsion, a similar result is no longer true: in the infinite dihedral group $\langle c \rangle_2 * \langle d \rangle_2$, every power of the commutator $[c, d]$ is also a commutator.

Corollary 1. *Let $\mathcal{F} = \ast_{\alpha \in I} G_\alpha$ be the free product of groups G_α , $\alpha \in I$, that contain no nontrivial elements of order $\leq n$. Then an equality $[x_1, y_1] \dots [x_k, y_k] = z^n$, where $n \geq 2k$, in \mathcal{F} implies that z is conjugate to an element of a free factor of \mathcal{F} .*

We remark that Corollary 1 subsumes the main result of Chen [2] on the stable commutator length of elements of free products of torsion free groups.

We now state more corollaries of Theorem.

Corollary 2. *Let \mathcal{F} be the free product of torsion free groups G_α , $\alpha \in I$. Suppose that*

$$c_1 \dots c_k d_1 \dots d_\ell = h_1 \dots h_m,$$

in \mathcal{F} , where $c_1, \dots, c_k, d_1, \dots, d_\ell, h_1, \dots, h_m$ are elements of \mathcal{F} such that c_i are commutators, d_i are conjugate to elements of free factors of \mathcal{F} , and h_i are conjugate to each other and are not conjugate to elements of free factors of \mathcal{F} . Then no element occurs in the sequence h_1, \dots, h_m more than $2k + \ell - 1$ times.

Corollary 3. *Suppose F is a free group and $c = h_1 \dots h_m \neq 1$ in F , where $c, h_1, \dots, h_m \in F$ are such that c is a commutator and h_1, \dots, h_m are conjugate to each other. Then h_1, \dots, h_m are all distinct.*

More generally, suppose that $c_1 \dots c_k = h_1 \dots h_m \neq 1$ in a free group F , where $c_1, \dots, c_k, h_1, \dots, h_m \in F$ are such that c_1, \dots, c_k are commutators and h_1, \dots, h_m are conjugate to each other. Then no element occurs in the sequence h_1, \dots, h_m more than $2k - 1$ times.

Corollary 4. *Let $A * B$ be the free product of torsion free groups A, B . Then no nontrivial element $a \in A$ is a product of elements that are conjugate to each other and are not conjugate to an element of A .*

*More generally, suppose that $a_1 b_1 \dots a_\ell b_\ell = h_1 \dots h_m \neq 1$ in $A * B$, where $a_1, \dots, a_\ell \in A \setminus \{1\}$, $b_1, \dots, b_\ell \in B \setminus \{1\}$, and $h_1, \dots, h_m \in A * B$ are conjugate to each other and are not conjugate to an element of $A \cup B$. Then no element occurs in the sequence h_1, \dots, h_m more than $2\ell - 1$ times.*

Let w be an element of a free product \mathcal{F} of groups G_α , $\alpha \in I$. A *mixed commutator factorization* for w is an equality in \mathcal{F} of the form

$$w = c_1 \dots c_k d_1 \dots d_\ell, \tag{1}$$

where c_i are commutators and d_j are conjugate to elements of free factors of \mathcal{F} . The *mixed genus* $\mathbf{mg}(w)$ of w is defined to be a minimal integer s such that $s = 2k + \ell$ over all mixed commutator factorizations (1) for w .

For example, if $\mathbf{mg}(w) \leq 1$ then w is conjugate to an element of a free factor of \mathcal{F} and if $\mathbf{mg}(w) = 2$ then w is a commutator or a product of two elements conjugate to nontrivial elements of free factors of \mathcal{F} .

We remark that Culler [8] introduced the genus $g(w)$ for an element w of a free product $A * B$ of two groups A, B as a minimal number of commutators needed to write w as the product of these commutators or $g(w) := \infty$ if w is not a product of commutators. Culler [8] gave an algorithm that computes the genus $g(w)$ of w whenever the genera of elements can be computed in free factors A, B . The genus $g(w)$ can be defined in the same fashion for an element w of an arbitrary free product \mathcal{F} of groups.

Let a free group F be considered as a free product of its cyclic subgroups. Grigorchuk and Kurchanov [14] defined the width $h(w)$ of an element w of F as a minimal number of elements that are conjugate to elements of free factors of F and that are needed to write w as their product. Grigorchuk and Kurchanov [14] gave an algorithm that computes the width $h(w)$ of $w \in F$, see also [17], [30]. The width $h(w)$ can be defined in the same manner for an element w of an arbitrary free product \mathcal{F} of groups.

It is worthwhile to note that our definition of the mixed genus $\mathbf{mg}(w)$ of an element w of an arbitrary free product \mathcal{F} combines the foregoing two definitions and the number $\mathbf{mg}(w)$ satisfies the inequalities $\mathbf{mg}(w) \leq 2g(w)$ and $\mathbf{mg}(w) \leq h(w)$.

A *quasiperiodic factorization* for an element w of a free product \mathcal{F} of groups G_α , $\alpha \in I$, is an equality in \mathcal{F} of the form

$$w = h_1^{n_1} \dots h_m^{n_m}, \quad (2)$$

where h_1, \dots, h_m are conjugate to each other and are not conjugate to an element of a free factor G_α and $m \geq 1$. The *quasiperiodicity* $\mathbf{qp}(w)$ of w is defined to be a maximal integer r such that $r = \sum_{j=1}^m (n_j - 1)$ over all quasiperiodic factorizations (2) for w if there are such factorizations and the set of such r is bounded above. If the set of such r is not bounded above, we set $\mathbf{qp}(w) := \infty$ and if there are no such factorizations for w , we set $\mathbf{qp}(w) := -\infty$.

It is clear that, for every $w \in \mathcal{F}$ such that w is not conjugate to an element of a free factor, we have $\mathbf{qp}(w) \geq 0$. As another example, consider two elements $u, v \in \mathcal{F}$ that are conjugate and are not conjugate to an element of a free factor of \mathcal{F} . Then $\mathbf{qp}(u^4 v^2) \geq 4$ and $\mathbf{qp}(u^3 v u v) \geq 3$ as $u^3 v u v = u^4 v^u v$, where $v^u := u^{-1} v u$.

Note that if $w = h_1 \dots h_m$ in \mathcal{F} , where h_1, \dots, h_m are conjugate to each other and are not conjugate to an element of a free factor G_α , and s elements among h_1, \dots, h_m are equal each other, then $\mathbf{qp}(w) \geq s - 1$. Indeed, we can apply the identity $uv = vu^v$ and rearrange the factors h_1, \dots, h_m in such a way that the equal s factors would form an s th power. This observation, in particular, implies that, if a free product $A * B$ has torsion, then $\mathbf{qp}(1) = \infty$. Indeed, if an element $a \in A$ has order $m > 1$ and $b \in B$ is nontrivial then

$$1 = [a, b][a, b]^{a^{-1}} \dots [a, b]^{a^{-m}} = ([a, b][a, b]^{a^{-1}} \dots [a, b]^{a^{-m}})^{2016}.$$

These equalities mean that $\mathbf{qp}(1) = \mathbf{qp}([a, b]) = \infty$. (It is not clear what could be $\mathbf{qp}(a), \mathbf{qp}(ab)$ in this situation.) On the other hand, for free products of groups without torsion we have a nicer situation.

Corollary 5. *Let \mathcal{F} be the free product of torsion free groups G_α , $\alpha \in I$. Then, for every $w \in \mathcal{F}$, the quasiperiodicity $\mathbf{qp}(w)$ of w satisfies $\mathbf{qp}(w) \leq \mathbf{mg}(w) - 2 < \infty$. Furthermore, $\mathbf{qp}(w)$ is finite if and only if w is not conjugate to an element of a free factor of \mathcal{F} .*

We remark that the bound $\mathbf{qp}(w) \leq \mathbf{mg}(w) - 2$ of Corollary 5 is sharp as follows from the equality

$$(ab)^n = a^n b^{a^{n-1}} b^{a^{n-2}} \dots b^a b$$

that proves that if $a \in A$, $b \in B$ are nontrivial then $\mathbf{qp}((ab)^n) \geq n - 1$ and $\mathbf{mg}((ab)^n) \leq n + 1$. The sharpness of the bound $\mathbf{qp}(w) \leq \mathbf{mg}(w) - 2$ also follows from Culler's [8] observation that $[a, b]^n$ is a product of k commutators whenever $n \leq 2k - 1$.

Our arguments utilize diagrams over free products of groups and are based on a car-crash lemma of [19], [20], [21], see also [10], that has had quite a few applications in group theory, see [3], [4], [5], [11], [12], [13], [18], [22], [23], [24], [25], [26], [27].

In Sect. 2, we define diagrams over free products of groups and prove a lemma on geometric meaning of the mixed genus. In Sect. 3, we state a car-crash lemma. Sect. 4 contains the proof of our Theorem.

2. PRELIMINARIES

Suppose that S is an oriented compact closed surface. Note that S need not be connected.

A map on S is a finite 2-complex Δ embedded into S . We call S the *underlying surface* for Δ , denoted $S = S(\Delta)$. If the embedding of Δ into S is surjective, i.e., Δ has no boundary, we say that the map Δ is *closed*.

The set of i -cells of a finite 2-complex Δ is denoted $\Delta(i)$, $i = 0, 1, 2$. The closures of i -cells of Δ for $i = 0, 1, 2$ are called *vertices*, *edges*, *faces*, resp. The 1-skeleton of Δ , consisting of vertices and edges, is a graph denoted $\Delta[1]$.

If F is a face of a map Δ then a boundary path ∂F of F is oriented in positive, i.e., in counterclockwise, direction. Recall that $S(\Delta)$ is oriented. If $\partial F = e_1 e_2 \dots e_k$, where e_1, e_2, \dots, e_k are oriented edges, then the subpaths $e_1 e_2, e_2 e_3, \dots, e_k e_1$ of ∂F are called *corners* of F . If $e_i e_{i+1}$ is a corner of F then the terminal vertex of e_i is called the *vertex* of $e_i e_{i+1}$ and is denoted $\nu(e_i e_{i+1})$.

If e is an oriented edge of a 2-complex Δ then e_-, e_+ denote the initial, terminal, resp., vertices of e . By e^{-1} we mean the edge with opposite to e orientation. If $p = e_1 \dots e_k$ is a path in Δ , where e_1, \dots, e_k are oriented edges, then the initial and terminal vertices of p are defined by $p_- := (e_1)_-$ and $p_+ := (e_k)_+$, resp., and $p^{-1} := e_k^{-1} \dots e_1^{-1}$.

Let $C(\Delta)$ denote the set of all corners of faces of a map Δ and let $A * B$ be a free product of two nontrivial groups A, B , where $A \cap B = \{1\}$. A map Δ is called a *diagram* over $A * B$ if Δ is equipped with two labeling functions

$$\varphi : C(\Delta) \rightarrow A \cup B, \quad \theta : \Delta(0) \rightarrow \{A, B\}$$

and the following conditions are satisfied.

- (D1) If u, v are two vertices of Δ connected by an edge then $\theta(u) \neq \theta(v)$. In particular, $\Delta[1]$ is a bipartite graph.
- (D2) For every corner $ee' \in C(\Delta)$ such that $\theta(\nu(ee')) = A$, we have $\varphi(ee') \in A$ and, for every corner $ee' \in C(\Delta)$ such that $\theta(\nu(ee')) = B$, we have $\varphi(ee') \in B$.

We remark that our definition of a diagram over $A * B$ is different from the definitions of diagrams over free products of groups used in books [28], [29] and is similar to the definition introduced in Howie's articles [15], [16].

Let F be a face of a diagram Δ over $A * B$ and let $\partial F = e_1 e_2 \dots e_k$, where e_1, e_2, \dots, e_k are oriented edges, be a boundary path of F . A *label* $\varphi(\partial F)$ of F is defined by setting

$$\varphi(\partial F) := \varphi(e_1 e_2) \varphi(e_2 e_3) \dots \varphi(e_{k-1} e_k),$$

i.e., $\varphi(\partial F)$ is the product of consecutive, in positive direction, φ -labels of corners of F . It is clear that $\varphi(\partial F)$ is a word over the alphabet $A \cup B$ and $\varphi(\partial F)$ is defined up to a cyclic permutation.

Let $p = e_i e_{i+1} \dots e_{i+\ell}$ be a subpath of a boundary path ∂F of a face F , where indices are modulo $k = |\partial F|$. We define the *label* $\varphi(p)$ of p to be the word

$$\varphi(p) := \varphi(e_i e_{i+1}) \varphi(e_{i+1} e_{i+2}) \dots \varphi(e_{i+\ell-1} e_{i+\ell}).$$

If $v \in \Delta(0)$ is a vertex in the interior of a diagram Δ over $A * B$, i.e., $v \notin \partial \Delta$, then a *label* $\varphi(v)$ of v is the product of φ -labels of consecutive, in negative direction, corners whose vertex is v . We say that v is an *A-vertex* if $\theta(v) = A$ and v is a *B-vertex* if $\theta(v) = B$. It is clear from the definitions that $\varphi(v) \in A$ if v is an *A-vertex* and $\varphi(v) \in B$ if v is a *B-vertex*. It is also clear that $\varphi(v)$ is defined up to conjugation in A or B . If $\varphi(v)$ is defined and $\varphi(v) = 1$ in A or in B , depending on type of v , then we say that v is a *regular* vertex. If $\varphi(v)$ is defined and $\varphi(v) \neq 1$ in A or B then v is called an *irregular* vertex. Note that a label $\varphi(v)$ is not defined for a vertex v on the boundary $\partial \Delta$ of Δ .

We remark that similar diagrams were considered in [15], [16], [19], [27] and some other papers but our definitions are slightly different.

For example, the diagram depicted in Fig. 1 has a torus as the underlying surface and it is drawn as a rectangle with opposite sides to be identified. This diagram contains two vertices, three edges, one face, and three corners with φ -label $a \in A$ and three corners with φ -label $b \in B$. If the vertices are regular, then $a^3 = 1$ in A and $b^3 = 1$ in B . The label of the face is $(ab)^3$. This diagram demonstrates that if $a \in A$ and $b \in B$ have order 3 then $(ab)^3$ is a commutator. A complete description of commutators in a free product of groups that are not conjugate to elements of free factors and are proper powers is given in [7].

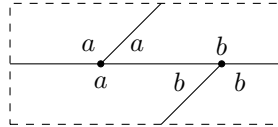


Fig. 1

We call a diagram Δ over $A * B$ *reduced* if Δ has no face with a corner whose φ -label is 1.

The *extended genus* $\text{eg}(\Delta)$ of a diagram Δ over $A * B$ is defined by

$$\text{eg}(\Delta) := 2 - \chi(\Delta) + r_0(\Delta), \quad (3)$$

where $\chi(\Delta) = |\Delta(0)| - |\Delta(1)| + |\Delta(2)|$ is the Euler characteristic of Δ and $r_0(\Delta)$ is the number of irregular vertices in Δ .

We consider elements of the free product $A * B$ as words over the alphabet $A \cup B$, where $A \cap B = \{1\}$, whose elements are called letters. A word $w = a_1 \dots a_\ell$, where $a_1, \dots, a_\ell \in A \cup B$ are letters, is called *reduced* if w is nonempty, none of the letters a_1, \dots, a_ℓ is 1 and, for every i , the letters a_i, a_{i+1} do not belong to the same free factor of $A * B$. The *length* of a word w is the number of letters in w , denoted $|w|$. A word w is *cyclically reduced* if w is nonempty and w^2 is reduced.

If u, w are two words over $A \cup B$, then $u \equiv w$ means the literal or letter-by-letter equality of words. If words u, w are equal as elements of $A * B$, we write $u \stackrel{*}{=} w$.

If w is a word over $A \cup B$, we let $\delta_1(w)$ denote the word obtained from w by deletion of all occurrences of the letter $1 \in A \cup B$. By writing $w \equiv_1 u$ we mean that $\delta_1(w) \equiv \delta_1(u)$. If $a, b \in A$ then by writing $a \stackrel{A}{=} b$ we mean that $a = b$ in A . Similarly, the notation $a \stackrel{B}{=} b$ means that $a, b \in B$ and $a = b$ in B .

We mention without proof that the mixed genus $\text{mg}(w)$ of a cyclically reduced word w over $A \cup B$ is equal to the extended genus $\text{eg}(\Delta)$ of a reduced closed diagram Δ over $A * B$ that contains a single face whose label is the word w . Here we need only the inequality $\text{mg}(w) \geq \text{eg}(\Delta)$ that follows from a more general lemma that we now state and prove.

Lemma 1. *Suppose that u_1, \dots, u_m are nonempty cyclically reduced words over the alphabet $A \cup B$. Then the minimal mixed genus $\text{mg}(w)$ of an element $w \in A * B$ of the form $w \stackrel{*}{=} s_1 u_1 s_1^{-1} \dots s_m u_m s_m^{-1}$ satisfies $\text{mg}(w) \geq \text{eg}(\Delta)$, where $\text{eg}(\Delta)$ is the minimal extended genus $\text{eg}(\Delta)$ of a reduced closed diagram Δ over $A * B$ that contains precisely m faces whose labels are the words u_1, \dots, u_m .*

Proof. Suppose that w_0 is a word of a minimal mixed genus $n = \text{mg}(w_0)$ among all words w of the form

$$w \equiv s_1 u_1 s_1^{-1} \dots s_m u_m s_m^{-1},$$

where each $s_i \in A * B$ is a reduced word or $s_i \equiv 1$. Since $n = \text{mg}(w_0)$, there is a factorization for w_0 of the form

$$w_0 \stackrel{*}{=} [v_1, t_1] \dots [v_k, t_k] d_1 \dots d_\ell,$$

where $n = 2k + \ell$, for every i , v_i, t_i are reduced words, and, for each j , $d_j \equiv d_{j,1} d_{j,0} d_{j,1}^{-1}$, $d_{j,0} \in A \cup B$, $d_{j,0} \neq 1$, $d_{j,1}$ is a reduced word or $d_{j,1} \equiv 1$.

Consider the word

$$s_m u_m^{-1} s_m^{-1} \dots s_1 u_1^{-1} s_1^{-1} v_1^{-1} t_1^{-1} v_1 t_1 \dots v_k^{-1} t_k^{-1} v_k t_k d_1 \dots d_\ell. \quad (4)$$

Let Δ_0 be a diagram over $A * B$ that consists of a single face H whose boundary path ∂H has the following factorization

$$\partial H = p_1 q_1 p_2 q_2 \dots p_L q_L,$$

where $L = 3m + 4k + \ell$, $p_1, q_1, \dots, p_L, q_L$ are subpaths of ∂H , and the sequence of words $\varphi(p_1), \dots, \varphi(p_L)$ is identical to the sequence of subwords $s_1, u_1^{-1}, s_1^{-1}, \dots, d_\ell$ distinguished in the word (4). The paths q_1, \dots, q_L have labels equal to powers of the letter $1 \in A \cup B$ and $|q_i| = 2$ or $|q_i| = 3$, hence, $\varphi(q_i) \equiv 1^{|q_i|-1}$ with $|q_i| - 1 \geq 1$.

The φ -label of a corner of H whose vertex is $(p_i)_-$ or $(p_i)_+$ is also 1. It is easy to see that we can assign θ -labels to corners of H so that both properties (D1)–(D2) hold true and Δ_0 is indeed a diagram over $A * B$. Note that the choice between $|q_i| = 2$ or $|q_i| = 3$ depends on $\theta((p_i)_+)$ and $\theta((p_{i+1})_-)$, here indices are modulo L .

Let x be one of the subwords $s_m, u_m^{-1}, s_m^{-1}, \dots, d_\ell$ distinguished in the word (4) and let $p(x)$ denote the corresponding path among p_1, p_2, \dots, p_L such that $\varphi(p(x)) \equiv x$, i.e., we assume that $p(s_m) = p_1, p(u_m^{-1}) = p_2, \dots, p(d_\ell) = p_L$.

We now make some surgeries over Δ_0 . We remark that θ -labels of vertices never change under these surgeries.

Observe that the subpath $p(d_j)$ of ∂H has even length because $d_j \equiv d_{j,1}d_{j,0}d_{j,1}^{-1}$, where $d_{j,0} \in A \cup B$ and $d_{j,0} \neq 1$, $d_{j,1}$ is a reduced word or $d_{j,1} \equiv 1$. Hence, there is a factorization $p(d_j) = p_1(d_j)p_2(d_j)$, where $|p_1(d_j)| = |p_2(d_j)|$ and $\varphi(p_1(d_j)) \equiv \varphi(p_2(d_j))^{-1}$. Therefore, by identifying $p_1(d_j)$ and $p_2(d_j)^{-1}$ for each $j = 1, \dots, \ell$, within H , so that the subpath $p(d_j)$ of ∂H turns into $\bar{p}_1(d_j)\bar{p}_1(d_j)^{-1}$, see Fig. 2, we obtain a diagram Δ_1 over $A * B$ with a single face, still denoted H , and ℓ irregular vertices $(\bar{p}_1(d_j))_+, j = 1, \dots, \ell$. Note that all vertices of the paths $\bar{p}_1(d_j)$, except for their end vertices $(\bar{p}_1(d_j))_- \in \partial\Delta_1$ and $(\bar{p}_1(d_j))_+$ are regular.

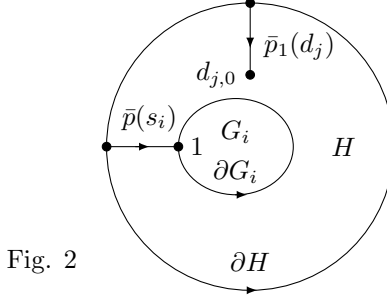


Fig. 2

Our next step is to identify, for every $i = 1, \dots, m$, the path $p(s_i)$ with $p(s_i^{-1})^{-1}$ within H , getting thereby a new path $\bar{p}(s_i)$ and a new map Δ_2 with $m + 1$ faces H, G_1, \dots, G_m such that the boundary path $\partial|_{(\bar{p}(s_i))_+} G_i$ of G_i starting at the vertex $(\bar{p}(s_i))_+$ is a subpath of $(\partial H)^{-1}$, see Fig. 2. We assign 1 as the φ -label to the corner of G_i whose vertex is $(\bar{p}(s_i))_+$ and, to every other corner of G_i , we assign φ -label equal to the inverse of the φ -label of the corner of H with the same vertex. Recall that θ -labels of vertices do not change. Such assignments produce a diagram Δ_2 over $A * B$ without additional irregular vertices because all vertices of ∂G_i and $\bar{p}(s_i)$ are regular.

We now identify the path $p(u_i^{-1})$ with $p(u_i)^{-1}$ and the path $p(t_i^{-1})$ with $p(t_i)^{-1}$ for every $i = 1, \dots, k$. Doing these identifications, results in a diagram Δ_3 over $A * B$ on an oriented surface of genus k such that Δ_3 consists of $m + 1$ faces H, G_1, \dots, G_m , Δ_3 has a single boundary component, denoted $\partial\Delta_3$, and Δ_3 contains ℓ irregular vertices. The images of paths $p(u_i), p(t_i)$ in Δ_3 are denoted $\bar{p}(u_i), \bar{p}(t_i)$, resp., $i = 1, \dots, k$.

Note that the vertices of paths $\bar{p}(u_i), \bar{p}(t_i)$, different from their end vertices, are all regular and the end vertices of $\bar{p}(u_i), \bar{p}(t_i)$ belong to the boundary path $\partial\Delta_3$. We also observe that if $ee' \in C(\Delta_3)$ is a corner whose vertex belongs to $\partial\Delta_3$ then $\varphi(ee') = 1$. Therefore, we may attach a new face G_0 such that $|\partial G_0| = |\partial\Delta_3|$ and $\varphi(\partial G_0) \equiv 1^{|\partial\Delta_3|}$ to Δ_3 by identifying the paths ∂G and $\partial\Delta_3$.

This attachment of G_0 to Δ_3 produces a new diagram Δ_4 over $A * B$ such that Δ_4 is closed, $\chi(\Delta_4) = 2 - 2k$, Δ_4 has ℓ irregular vertices,

$$\text{eg}(\Delta_4) = 2k + \ell = \text{mg}(w_0),$$

Δ_4 contains m faces G_1, \dots, G_m such that $\varphi(\partial G_i) \equiv_1 u_i$, where $i = 1, \dots, m$, and Δ_4 contains two more faces H, G_0 such that $\varphi(\partial H) \equiv \varphi(\partial G_0) \equiv 1$.

Thus we have constructed a closed diagram Δ_4 over $A * B$ with some desired properties except for the properties of being reduced and having precisely m faces whose labels are the words u_1, \dots, u_m .

For a closed diagram Δ over $A * B$ consider the parameter

$$\tau(\Delta) := (-\chi(\Delta), r_0(\Delta), |\Delta(1)|),$$

where as above $\chi(\Delta)$ is the Euler characteristic of Δ , $r_0(\Delta)$ is the number of irregular vertices of Δ , and $|\Delta(1)|$ is the number of nonoriented edges of Δ . We partially order diagrams Δ over $A * B$ according to their parameters $\tau(\Delta)$ which are ordered lexicographically, i.e., $\tau(\Delta) < \tau(\Delta')$ if and only if $-\chi(\Delta) < -\chi(\Delta')$ or $-\chi(\Delta) = -\chi(\Delta')$ and $r_0(\Delta) < r_0(\Delta')$ or $-\chi(\Delta) = -\chi(\Delta')$ and $r_0(\Delta) = r_0(\Delta')$ and $|\Delta(1)| < |\Delta'(1)|$.

Initializing, we set $\tilde{\Delta} := \Delta_4$ and note that

$$\tau(\tilde{\Delta}) = (2 - 2k, \ell, \tilde{\Delta}(1)), \quad \text{eg}(\tilde{\Delta}) \leq \text{mg}(w_0).$$

In our inductive arguments below we do not assume that $\tilde{\Delta}$ is necessarily connected but we do assume that $\tilde{\Delta}$ has the following property.

- (P) Every connected component of a diagram Δ over $A * B$ contains a face F such that $\varphi(\partial F) \equiv_1 u_i$ for some $i = 1, \dots, m$.

Note that the number of connected components of a diagram Δ over $A * B$ with property (P) is at most m . Hence, $-\chi(\Delta) \geq -2m$ because $-\chi(\Delta) \geq -2$ whenever Δ is connected. Since the second and the third components of $\tau(\Delta)$ are nonnegative integers, it follows that there is no strictly decreasing infinite chain

$$\tau(\Delta_1) > \tau(\Delta_2) > \dots$$

in which diagrams $\Delta_1, \Delta_2, \dots$ have property (P). This means that we may use induction on parameter $\tau(\Delta)$ in our arguments below if intermediate diagrams, similarly to $\tilde{\Delta}$, also have property (P).

Now we will make more surgeries over $\tilde{\Delta}$ aimed to get a reduced diagram.

If $\tilde{\Delta}$ is reduced and has property (P) then $\tilde{\Delta}$ is a required diagram and our proof is complete.

Suppose that there is a corner ef of a face F of $\tilde{\Delta}$ such that $\varphi(ef) = 1$. Consider three possible cases.

Case 1: Assume that $e = f^{-1}$, i.e., the vertex $e_+ = f_-$ has degree 1 and the corner $ef = ee^{-1}$ is the only corner in $\tilde{\Delta}$ whose vertex is e_+ .

If the degree of the vertex e_- is also 1 then the connected component of $\tilde{\Delta}$ that contains e, f is a sphere that contains the single face F such that $|\partial F| = 2$ and $\varphi(\partial F) \equiv 1c$, where c is the φ -label of the second corner of F . Since u_1, \dots, u_m are cyclically reduced words, it follows that the label of F may not be one of u_1, \dots, u_m . This contradiction to property (P) of $\tilde{\Delta}$ proves that the degree of e_- is greater than 1. Hence, we may take the edges e, f out of $\tilde{\Delta}$ creating thereby a diagram $\tilde{\Delta}_1$ with

property (P) and $\text{eg}(\tilde{\Delta}_1) = \text{eg}(\tilde{\Delta})$. The two consecutive corners $e'e$, $e^{-1}f'$ of F will disappear and, in their place, we obtain a single corner $e'f'$ whose φ -label is defined by $\varphi(e'f') := \varphi(e'e)\varphi(ff')$, see Fig. 3, where $\varphi(e'e) = a_1$, $\varphi(ff') = a_2$ and $a_1, \dots, a_4 \in A \cup B$. In view of inequality $\tau(\tilde{\Delta}_1) < \tau(\tilde{\Delta})$ and $\text{eg}(\tilde{\Delta}_1) = \text{eg}(\tilde{\Delta})$, we can use the induction hypothesis and Case 1 is complete.

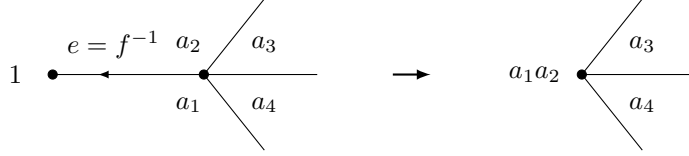


Fig. 3

Case 2: Suppose $e_- \neq f_+$.

In this case we fold the edges e and f^{-1} within F , i.e., we identify e and f^{-1} through the “corner” of F between them. The vertices e_- , f_+ become identical and the two corners $e'e$, ff' of F , whose vertices were e_- , f_+ before the fold, turn into a single corner $e'f'$ whose φ -label is defined by $\varphi(e'f') := \varphi(e'e)\varphi(ff')$, see Fig. 4, where $\varphi(e'e) = a_1$, $\varphi(ff') = a_5$ and $a_1, \dots, a_6 \in A \cup B$. As a result, we obtain a diagram $\tilde{\Delta}_1$ over $A * B$ such that $\tau(\tilde{\Delta}_1) < \tau(\tilde{\Delta})$ and $\text{eg}(\tilde{\Delta}_1) = \text{eg}(\tilde{\Delta})$. By the induction hypothesis, Case 2 is complete.

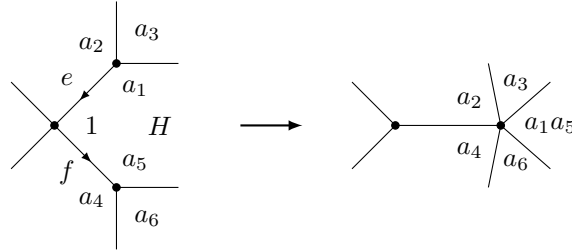


Fig. 4

Case 3: Suppose $e \neq f^{-1}$ and $e_- = f_+$.

In this case, the path ef is closed and defines a simple closed curve on the underlying surface $S(\tilde{\Delta})$. We cut $\tilde{\Delta}$ along this curve and obtain a new diagram $\tilde{\Delta}_0$ with two boundary components, oriented clockwise, which we denote by $e'f'$ and $(e''f'')^{-1}$, where e' , e'' are the images of e in $\tilde{\Delta}_0$, f' , f'' are the images of f in $\tilde{\Delta}_0$, and $e'f'$ is the image of the corner ef of H in $\tilde{\Delta}_0$, see Fig. 5.

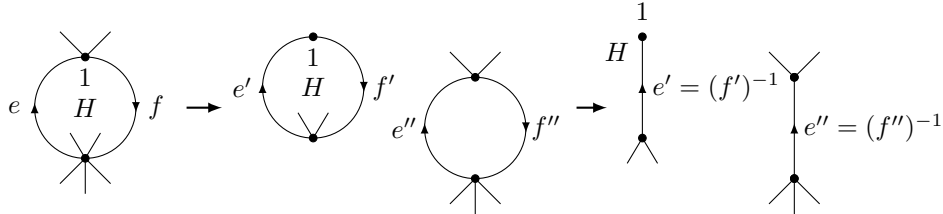


Fig. 5

Note that $\chi(\tilde{\Delta}_0) = \chi(\tilde{\Delta})$ and the closed paths $e'f'$, $e''f''$ might belong to different connected components of $\tilde{\Delta}_0$ which happens when ef defines a separating curve on $S(\tilde{\Delta})$.

We identify the edges e' and $(f')^{-1}$ and the edges e'' and $(f'')^{-1}$ thus eliminating the boundary of $\tilde{\Delta}_0$. The result is a closed diagram $\tilde{\Delta}_1$ over $A * B$ such that

$$\chi(\tilde{\Delta}_1) = \chi(\tilde{\Delta}_0) + 2 = \chi(\tilde{\Delta}) + 2.$$

Observe that the images of vertices e'_-, e''_- in $\tilde{\Delta}_1$ could be both irregular even if e_- is regular in $\tilde{\Delta}$, the image of e'_+ in $\tilde{\Delta}_1$ is regular and the image of e''_+ in $\tilde{\Delta}_1$ is regular if and only if e_+ is regular in $\tilde{\Delta}$. This means that $r_0(\tilde{\Delta}_1) \leq r_0(\tilde{\Delta}) + 2$. Therefore, we have

$$\text{eg}(\tilde{\Delta}_1) = 2 - \chi(\tilde{\Delta}_1) + r_0(\tilde{\Delta}_1) \leq \text{eg}(\tilde{\Delta}). \quad (5)$$

Note that $\tilde{\Delta}_1$ might have a connected component $\tilde{\Delta}_{1,1}$ with the property that $\varphi(\partial G) \equiv 1$ for every face G in $\tilde{\Delta}_{1,1}$, i.e., $\tilde{\Delta}_1$ might lack the property (P). Since $\tilde{\Delta}$ has property (P), it follows that there is at most one such component $\tilde{\Delta}_{1,1}$ in $\tilde{\Delta}_1$. If $\tilde{\Delta}_{1,1}$ does exist then we take $\tilde{\Delta}_{1,1}$ out of $\tilde{\Delta}_1$ and denote thus obtained diagram $\tilde{\Delta}_2$. If $\tilde{\Delta}_{1,1}$ does not exist then we set $\tilde{\Delta}_2 := \tilde{\Delta}_1$. Clearly, $\tilde{\Delta}_2$ has property (P).

First we consider the subcase when either $\tilde{\Delta}_{1,1}$ does not exist or $\tilde{\Delta}_{1,1}$ exists and $\chi(\tilde{\Delta}_{1,1}) \leq 0$. Since $\chi(\tilde{\Delta}_1) = \chi(\tilde{\Delta}_2) + \chi(\tilde{\Delta}_{1,1})$, it follows from the definitions and the inequality (5) that $-\chi(\tilde{\Delta}_2) \leq -\chi(\tilde{\Delta}) - 2$ and $\text{eg}(\tilde{\Delta}_2) \leq \text{eg}(\tilde{\Delta})$. Hence, $\tau(\tilde{\Delta}_2) < \tau(\tilde{\Delta})$ and, by the induction hypothesis, this subcase is complete.

Now assume that $\tilde{\Delta}_{1,1}$ exists and $\chi(\tilde{\Delta}_{1,1}) > 0$. Since $\tilde{\Delta}_{1,1}$ is oriented and connected, it follows that $\chi(\tilde{\Delta}_{1,1}) = 2$ and $\tilde{\Delta}_{1,1}$ is a sphere. Since $\chi(\tilde{\Delta}_1) = \chi(\tilde{\Delta}_2) + \chi(\tilde{\Delta}_{1,1})$, we have $\chi(\tilde{\Delta}_2) = \chi(\tilde{\Delta})$.

Let us show that $r_0(\tilde{\Delta}_2) \leq r_0(\tilde{\Delta})$. It follows from our construction that either $r_0(\tilde{\Delta}_1) = r_0(\tilde{\Delta})$ or $r_0(\tilde{\Delta}_1) = r_0(\tilde{\Delta}) + 2$. If $r_0(\tilde{\Delta}_1) = r_0(\tilde{\Delta})$ then $r_0(\tilde{\Delta}_2) \leq r_0(\tilde{\Delta})$ as desired. Assume that $r_0(\tilde{\Delta}_1) = r_0(\tilde{\Delta}) + 2$. Then it follows from the definitions that $r_0(\tilde{\Delta}_{1,1}) \geq 1$ because the image of the vertex e'_- in $\tilde{\Delta}_{1,1}$ is irregular. It is not difficult to show (e.g., by induction on $(|\tilde{\Delta}_{1,1}(2)|, |\tilde{\Delta}_{1,1}(1)|)$) that the equality $r_0(\tilde{\Delta}_{1,1}) = 1$ is impossible. Therefore, $r_0(\tilde{\Delta}_{1,1}) \geq 2$ and we can conclude that

$$r_0(\tilde{\Delta}_2) \leq r_0(\tilde{\Delta}_1) - 2 \leq r_0(\tilde{\Delta}),$$

as desired.

Since $|\tilde{\Delta}_2(1)| + |\tilde{\Delta}_{1,1}(1)| = |\tilde{\Delta}(1)|$ and $|\tilde{\Delta}_{1,1}(1)| > 0$, it follows from $\chi(\tilde{\Delta}_2) = \chi(\tilde{\Delta})$ and $r_0(\tilde{\Delta}_2) \leq r_0(\tilde{\Delta})$ that $\tau(\tilde{\Delta}_2) < \tau(\tilde{\Delta})$. It is also clear that $\text{eg}(\tilde{\Delta}_2) \leq \text{eg}(\tilde{\Delta})$, hence, by the induction hypothesis, Case 3 is complete.

Thus in all Cases 1–3 we have been able to construct a diagram $\tilde{\Delta}'$ over $A * B$ such that $\text{eg}(\tilde{\Delta}') \leq \text{eg}(\tilde{\Delta})$, $\tau(\tilde{\Delta}') < \tau(\tilde{\Delta})$ and $\tilde{\Delta}'$ contains m faces G_1, \dots, G_m such that $\varphi(\partial G_i) \equiv_1 u_i$, $i = 1, \dots, m$. This completes the proof of Lemma 1. \square

3. CAR MOTIONS

This Section is similar to a corresponding section of [12] and contains necessary definitions and statements of [20], [21] with some simplifications.

Consider a closed map Δ on a closed oriented compact surface. A *car* moving around a face F of Δ is an orientation preserving covering of the boundary path ∂F of F by an oriented circle $C = \mathbb{R}/M\mathbb{Z}$, called the *circle of time* and regarded as the quotient of the real numbers \mathbb{R} by its subgroup $M\mathbb{Z}$, where \mathbb{Z} is the set of integers and $M \in \mathbb{R}$.

Informally, a car is a point moving along the boundary path of a face in counterclockwise direction (the interior of the face remains on the left) without U-turns and stops. The motion is periodic.

The *degree* of a vertex v of a map Δ is the number of oriented edges of Δ whose terminal vertex is v . By the definition, a point in the interior of an edge of Δ has *degree* two.

Let v be a point of the 1-skeleton $\Delta[1]$ of Δ and suppose that the number of cars being at a moment of time t at the point v is equal to the degree of v . Then v is called a *point of complete collision*.

A *multiple car motion of period T* on Δ is a set of cars $\alpha_{F,j} : C \rightarrow \partial F$, defined for every face F of Δ and for every $j = 1, \dots, d_F$, where $d_F \geq 1$ is an integer, such that the following hold true.

- (M1) If $d_F > 1$ then $\alpha_{F,j}(t+T) = \alpha_{F,j+1}(t)$ for every $t \in \mathbb{R}$ and $j \in \{1, \dots, d_F\}$, here the second subscripts are modulo d_F and addition of points of C is defined according to $C = \mathbb{R}/M\mathbb{Z}$, where M is an integer multiple of T .
- (M2) For every face F of Δ , there exists a partition of ∂F into d_F consecutive arcs with disjoint interiors such that, during the time interval $[0, T]$, each car $\alpha_{F,j}$ is moving along the j th arc of the partition.

Lemma 2 ([20], [21]). *For every multiple car motion defined on a closed map Δ on an oriented compact surface, the number of points of complete collision is at least*

$$\chi(\Delta) + \sum_{F \in \Delta(2)} (d_F - 1),$$

where the summation runs over all faces F of Δ .

We remark that, in articles [20], [21], Lemma 2 is stated and proved for connected surfaces, but it remains valid in nonconnected case because both parts of the inequality in Lemma 2 are additive under disjoint union.

4. PROOF OF THEOREM

Note that an arbitrary free product $\mathcal{F} = \ast_{\alpha \in I} G_\alpha$ of nontrivial groups G_α , where $|I| > 1$, can be embedded into a free product $A \ast B$ of two groups A, B by means of a monomorphism $\mu : \mathcal{F} \rightarrow A \ast B$ in such a way that properties (E1)–(E2) hold true.

- (E1) The set of orders of elements of $A \ast B$ is identical to that of \mathcal{F} .
- (E2) An element $w \in \mathcal{F}$ is conjugate in \mathcal{F} to an element of a free factor G_α if and only if $\mu(w)$ is conjugate in $A \ast B$ to an element of $A \cup B$.

Indeed, let $A := \mathcal{F}$ and let $B := F(b_\alpha; \alpha \in I)$ be a free group whose free generators are letters $b_\alpha, \alpha \in I$. Then a required embedding $\mu : \mathcal{F} \rightarrow A \ast B$ can be defined by setting $\mu(g) := b_\alpha^{-1} g b_\alpha$ for every $g \in G_\alpha$.

Observe that if $w \in \mathcal{F}$ then it follows from property (E2) that

$$\text{qp}(w) \leq \text{qp}(\mu(w)) \quad \text{and} \quad \text{mg}(\mu(w)) \leq \text{mg}(w).$$

Hence, in view of property (E1), it suffices to prove our Theorem for a free product $A * B$ of two factors A, B .

Let $w \in A * B$ be a word such that $\mathbf{qp}(w)$ is finite and consider a quasiperiodic factorization for w of the form

$$w = s_1 u^{n_1} s_1^{-1} s_2 u^{n_2} s_2^{-1} \dots s_m u^{n_m} s_m^{-1},$$

where u is a cyclically reduced word, $s_j \in A * B$, $n_j > 0$, and $\mathbf{qp}(w) = \sum_j (n_j - 1)$. By Lemma 1, there exists a reduced diagram Δ over $A * B$ such that Δ contains precisely m faces F_1, \dots, F_m whose labels are the words $u^{n_1}, u^{n_2}, \dots, u^{n_m}$, resp., and

$$\mathbf{eg}(\Delta) \leq \mathbf{mg}(w). \quad (6)$$

Denote

$$u \equiv a_1 b_1 \dots a_r b_r, \quad (7)$$

where $a_i \in A$, $b_i \in B$ and $a \neq 1$, $b \neq 1$.

We will now define a multiple car motion on Δ in the following manner. For every $j = 1, \dots, m$, there are n_j cars that move around the boundary path ∂F_j , where $\varphi(\partial F_j) \equiv u^{n_j}$, with constant speed, one edge per unit of time, and, at the initial moment of time, $t = 0$, the cars are located at distinct corners whose φ -labels are b_r , here b_r means the last letter of u , see (7). It is easy to see that this is a periodic motion with period $2r$. By Lemma 2, there are at least $\chi(\Delta) + \sum_j (n_j - 1)$ points of complete collision in Δ .

We will now analyze where these complete collisions may occur.

First, note that a complete collision may not occur at an interior point of an edge of Δ . Indeed, at every even moment of time $t = 2i$, where $i \in \mathbb{Z}$, all cars are located at B -vertices, while at every odd moment of time $t = 2i + 1$ all cars are located at A -vertices. Therefore, during the time interval $(2i, 2i + 1)$ every car is moving from a B -vertex to an A -vertex, while during the time interval $(2i - 1, 2i)$ every car is moving from an A -vertex to a B -vertex. Thus any two cars are never moving along the same edge in opposite directions and may not collide in the interior of an edge.

Second, observe that a complete collision may not occur at a regular vertex. To prove this claim, we note that at every integer moment of time all cars are located at corners with the same φ -label, as denoted in (7). More specifically, at an even moment of time $t = 2i$, where $i \in \mathbb{Z}$, all cars are located at corners with φ -label being b_i , as indicated in (7), here indices are modulo r , and, at an odd moment of time $t = 2i + 1$, all cars are located at corners with φ -label being a_i , as denoted in (7). Therefore, all corners, whose vertex v is a given point of a complete collision, must have the same φ -label, as indicated in the factorization (7). If v is a regular vertex of degree d then it follows from the definition of a regular vertex that $a_i^d \stackrel{A}{=} 1$ or $b_i^d \stackrel{B}{=} 1$ for some i . Since d does not exceed the number of all corners in Δ with φ -label a_i or b_i , it follows that $d \leq \sum_j n_j$. However, this inequality contradicts the assumption that free factors A, B have no nontrivial elements of order $\leq \sum_j n_j$. This contradiction proves our claim.

Therefore, complete collisions can only occur at irregular vertices of Δ . Recall that, by Lemma 2, there are at least $\chi(\Delta) + \sum_j (n_j - 1)$ points of complete collision in Δ . Hence, we conclude that the number of irregular vertices of Δ is at least

$\chi(\Delta) + \sum_j (n_j - 1)$, i.e.,

$$r_0(\Delta) \geq \chi(\Delta) + \sum_j (n_j - 1).$$

Therefore, it follows from the inequality (6) that

$$\text{qp}(w) = \sum_j (n_j - 1) \leq -\chi(\Delta) + r_0(\Delta) = \text{eg}(\Delta) - 2 \leq \text{mg}(w) - 2,$$

as required. This completes the proof of Theorem.

Corollaries are straightforward from the definitions and Theorem.

REFERENCES

- [1] G. Baumslag and A. Steinberg, *Residual nilpotence and relations in free groups*, Bull. Amer. Math. Soc. **70**(1964), 283–284.
- [2] L. Chen, *Spectral gap of scl in free products*, preprint, [arXiv:1611.07936 \[math.GT\]](#).
- [3] A. Clifford and R. Z. Goldstein, *Tesselations of S^2 and equations over torsion-free groups*, Proc. Edinburgh Math. Soc. **38**(1995), 485–493.
- [4] A. Clifford and R. Z. Goldstein, *Equations with torsion-free coefficients*, Proc. Edinburgh Math. Soc. **43**(2000), 295–307.
- [5] M. M. Cohen and C. Rourke, *The surjectivity problem for one-generator, one-relator extensions of torsion-free groups*, Geometry & Topology **5**(2001), 127–142.
- [6] J. A. Comerford, L. P. Comerford and C. C. Edmunds, *Powers as products of commutators*, Comm. Algebra **19**(1991), 675–684.
- [7] L. P. Comerford, C. C. Edmunds, and G. Rosenberger, *Commutators as powers in free products of groups*, Proc. Amer. Math. Soc. **122**(1994), 47–52.
- [8] M. Culler, *Using surfaces to solve equations in free groups*, Topology **20**(1981), 133–145.
- [9] A. J. Duncan and J. Howie, *The genus problem for one-relator products of locally indicable groups*, Math. Z. **208**(1991), 225–237.
- [10] R. Fenn and C. Rourke, *Klyachko’s methods and the solution of equations over torsion-free groups*, L’Enseignement Math. **42**(1996), 49–74.
- [11] R. Fenn and C. Rourke, *Characterisation of a class of equations with solution over torsion-free groups*, “The Epstein Birthday Schrift”, (ed. by I. Rivin, C. Rourke and C. Series), Geometry and Topology Monographs **1**(1998), 159–166.
- [12] E. V. Frenkel and Ant. A. Klyachko, *Commutators cannot be proper powers in metric small-cancellation torsion-free groups*, preprint, [arXiv:1210.7908 \[math.GR\]](#).
- [13] M. Forester and C. Rourke, *Diagrams and the second homotopy group*, Comm. Anal. Geom. **13**(2005), 801–820.
- [14] R. I. Grigorchuk and P. F. Kurchanov, *On the width of elements in free groups*, Ukrainian Math. J. **43**(1991), 911–918.
- [15] J. Howie, *The solution of length three equations over groups*, Proc. Edinburgh Math. Soc. **26**(1983), 89–96.
- [16] J. Howie, *The quotient of a free product of groups by a single high-powered relator. II. Fourth powers*, Proc. London Math. Soc. **61**(1990), 33–62.
- [17] S. V. Ivanov, *The bounded and precise word problems for presentations of groups*, preprint, [arXiv:1606.08036 \[math.GR\]](#).
- [18] S. V. Ivanov and Ant. A. Klyachko, *The asphericity and Freiheitssatz for certain LOT-presentations of groups*, Internat. J. Algebra Comp. **11**(2001), 291–300.
- [19] Ant. A. Klyachko, *A funny property of a sphere and equations over groups*, Comm. Algebra **21**(1993), 2555–2575.
- [20] Ant. A. Klyachko, *Asphericity tests*, Internat. J. Algebra Comp. **7**(1997), 415–431.
- [21] Ant. A. Klyachko, *The Kervaire–Laudenbach conjecture and presentations of simple groups*, Algebra and Logic **44**(2005), 219–242.
- [22] Ant. A. Klyachko, *How to generalize known results on equations over groups*, Mathematical Notes **79**(2006), 377–386.

- [23] Ant. A. Klyachko, *The SQ-universality of one-relator relative presentations*, Sbornik Mathematics **197**(2006), 1489–1508.
- [24] Ant. A. Klyachko, *Free subgroups of one-relator relative presentations*, Algebra and Logic **46**(2007), 158–162.
- [25] Ant. A. Klyachko, *The structure of one-relator relative presentations and their centres*, J. Group Theory **12**(2009), 923–947.
- [26] Ant. A. Klyachko and D. E. Lurye, *Relative hyperbolicity and similar properties of one-generator one-relator relative presentations with powered unimodular relator*, J. Pure Appl. Algebra **216**(2012), 524–534.
- [27] Le Thi Giang, *The relative hyperbolicity of one-relator relative presentations*, J. Group Theory **12**(2009), 949–959.
- [28] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.
- [29] A. Yu. Ol’shanskii, *Geometry of defining relations in groups*, Nauka, Moscow, 1989; English translation: *Math. and Its Applications, Soviet series*, vol. 70, Kluwer Acad. Publ., 1991.
- [30] A. Yu. Ol’shanskii, *On calculation of width in free groups*, London Math. Soc. Lecture Note Ser. **204**(1995), 255–258.
- [31] M. P. Schützenberger, *Sur l’équation $a^{2+n} = b^{2+m}c^{2+p}$ dans un groupe libre*, C. R. Acad. Sci. Paris **248**(1959), 2435–2436.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801, U.S.A.
E-mail address: `ivanov@illinois.edu`

DEPARTMENT OF MECHANICS AND MATHEMATICS, MOSCOW STATE UNIVERSITY, MOSCOW, 119991, RUSSIA
E-mail address: `klyachko@mech.math.msu.su`